

Representations of the Schrödinger-Virasoro algebras¹

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Abstract. In this paper it is proved that an irreducible weight module with finite-dimensional weight spaces over the Schrödinger-Virasoro algebras is a highest/lowest weight module or a uniformly bounded module. Furthermore, indecomposable modules of the intermediate series over these algebras are completely determined.

Key words: Schrödinger-Virasoro algebras, modules of the intermediate series, Harish-Chandra modules.

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1. Introduction

Let $s = 0$ or $\frac{1}{2}$. The *Schrödinger-Virasoro algebra* $\mathcal{L}[s]$ introduced in [1, 2, 3], in the context of non-equilibrium statistical physics as a by-product of the computation of n -point functions that are covariant under the action of the Schrödinger group, is the infinite-dimensional Lie algebra with \mathbb{C} -basis $\{L_m, Y_p, M_n, c \mid m, n \in \mathbb{Z}, p \in s + \mathbb{Z}\}$ and Lie brackets,

$$[L_m, L_{m'}] = (m' - m)L_{m'+m} + \delta_{m,-m'} \frac{m^3 - m}{12} c, \quad (1.1)$$

$$[L_m, Y_p] = (p - \frac{m}{2})Y_{p+m}, \quad (1.2)$$

$$[L_m, M_n] = nM_{n+m}, \quad (1.3)$$

$$[Y_p, Y_{p'}] = (p' - p)M_{p'+p}, \quad (1.4)$$

$$[Y_p, M_n] = [M_n, M_{n'}] = [\mathcal{L}, c] = 0. \quad (1.5)$$

The Lie algebra $\mathcal{L}[s]$ contains as subalgebras both the Lie algebra \mathcal{S} of invariance of the free Schrödinger equation and the central charge-free Virasoro algebra $\mathcal{V}ir$, where \mathcal{S} is the infinite dimensional Lie algebra, called the *Schrödinger algebra*, with the \mathbb{C} -basis $\{Y_p, M_n \mid n \in \mathbb{Z}, p \in s + \mathbb{Z}\}$, and $\mathcal{V}ir$ is the *Virasoro algebra* with the \mathbb{C} -basis $\{L_n, c \mid n \in \mathbb{Z}\}$. The Lie algebra $\mathcal{L}[\frac{1}{2}]$ is called the *original Schrödinger-Virasoro algebra*, while $\mathcal{L}[0]$ is called the *twisted Schrödinger-Virasoro algebra*.

It is well known that one attempt of introducing two-dimensional conformal field theory is to understand the universal behavior of two-dimensional statistical systems at equilibrium and at the critical temperature. A systematic investigation of the theory of representations

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of the Virasoro algebra in the 80's led to introduce the unitary minimal models, corresponding to the unitary highest weight representations of the Virasoro algebra with central charge less than one. Miraculously, covariance alone is enough to allow the computation of the n -point functions for these highly constrained physical models. Since both original and twisted Schrödinger-Virasoro Lie algebras are closely related to the Schrödinger Lie algebra and the Virasoro Lie algebra, it is highly expected that they should consequently play an important role akin to that of the Virasoro Lie algebra in two-dimensional equilibrium statistical physics (see, e.g., [1, 2, 3, 4, 6, 10]).

Partially due to the above-stated reasons, the Schrödinger-Virasoro Lie algebras have recently drawn some attentions in the literature. In particular, the sets of generators provided by the cohomology classes of the cocycles for both original and twisted Schrödinger-Virasoro Lie algebras were presented in [6], and the derivation algebra and the automorphism group of the twisted sector were determined in [4]. Furthermore, vertex algebra representations of the Schrödinger-Virasoro Lie algebras out of a charged symplectic boson and a free boson with its associated vertex operators were constructed in [10].

Our motivation in studying the Schrödinger-Virasoro Lie algebras is to have a better understanding of their representations. Let us formulate our main results below.

Denote

$$\mathcal{H} = \text{span}\{L_0, M_0, c\}, \quad (1.6)$$

which is a maximal torus of $\mathcal{L}[s]$ (note that in case of $s = 0$, since the adjoint operator ad_{Y_0} is not semi-simple, $\mathcal{H} \oplus \mathbb{C}Y_0$ is not a torus but a Cartan subalgebra of $\mathcal{L}[0]$). Denote by \mathcal{H}^* the dual space of \mathcal{H} .

Definition 1.1 A module V over $\mathcal{L}[s]$ is called a

- (i) *Harish-Chandra module* if V admits a finite-dimensional weight space decomposition $V = \bigoplus_{\lambda \in \mathcal{H}^*} V^\lambda$, where $V^\lambda = \{v \in V \mid xv = \lambda(x)v, x \in \mathcal{H}\}$ such that $\dim V^\lambda < \infty$ for all $\lambda \in \mathcal{H}^*$ (in case $V^\lambda \neq 0$, we call λ a *weight of V*);
- (ii) *uniformly bounded module* if it is a Harish-Chandra module such that there exists some $N > 0$ with $\dim V^\lambda \leq N$ for all $\lambda \in \mathcal{H}^*$;
- (iii) *module of the intermediate series* if V is an indecomposable Harish-Chandra module such that $\dim V^\lambda \leq 1$ for all $\lambda \in \mathcal{H}^*$.

Remark 1.2 (i) Since the Schrödinger subalgebra \mathcal{S} is an ideal of \mathcal{L} , if \mathcal{S} acts trivially on a module V , then V is simply a module over $\mathcal{V}ir$. Thus in the following, we always suppose that \mathcal{S} acts nontrivially on V .

- (ii) It is well known (see, e.g., [7, 8]) that a module V of the intermediate series over $\mathcal{V}ir$ is a quotient of one of the modules $A_{a,b}$, $A(\alpha)$, $B(\alpha)$ for some $a, b, \alpha \in \mathbb{C}$, they all have

the basis $\{x_k \mid k \in \mathbb{Z}\}$ such that c acts trivially, and for $n, k \in \mathbb{Z}$,

$$A_{a,b} : L_n x_k = (a + k + bn)x_{k+n}, \quad (1.7)$$

$$A(\alpha) : L_n x_k = (k + n)x_{k+n} \ (k \neq 0), \quad L_n x_0 = n(n + \alpha)x_n, \quad (1.8)$$

$$B(\alpha) : L_n x_k = kx_{k+n} \ (k \neq -n), \quad L_n x_{-n} = -n(n + \alpha)x_0. \quad (1.9)$$

The main results of this paper are presented in the following theorem.

Theorem 1.3 (i) *An irreducible Harish-Chandra module over $\mathcal{L}[s]$ is either a highest/lowest weight module or a uniformly bounded one.*

(ii) *A module of the intermediate series over $\mathcal{L}[0]$ is simply a module of the intermediate series over $\mathcal{V}ir$.*

(iii) *A module V of the intermediate series over $\mathcal{L}[\frac{1}{2}]$ such that \mathcal{S} acts nontrivially on V is one of the modules*

$$A_{a,b}, \ B_{a,b}, \ C_a, \ D_a, \ A_1(\alpha), \ A_2(\alpha), \ B_1(\alpha), \ B_2(\alpha), \ C(\alpha, \alpha'), \ D(\beta, \beta'),$$

or one of their quotients for $a, b, \alpha, \alpha', \beta, \beta' \in \mathbb{C}$, whose module structures are given as follows (the central element c acts as zero), where $k \in \frac{1}{2}\mathbb{Z}$, $i, n \in \mathbb{Z}$, $j, p \in \frac{1}{2} + \mathbb{Z}$,

$$A_{a,b} : M_n x_k = 0, \quad (1.10)$$

$$Y_p x_i = (a + i + 2bp)x_{i+p}, \quad Y_p x_j = 0, \quad (1.11)$$

$$L_n x_i = (a + i + bn)x_{i+n}, \quad L_n x_j = (a + j + (b + \frac{1}{2})n)x_{j+n}, \quad (1.12)$$

$$B_{a,b} : M_n x_k = 0, \quad (1.13)$$

$$Y_p x_i = 0, \quad Y_p x_j = x_{j+p}, \quad (1.14)$$

$$L_n x_i = (a + i + bn)x_{i+n}, \quad L_n x_j = (a + j + (b + \frac{1}{2})n)x_{j+n}, \quad (1.15)$$

$$C_a : M_n x_k = 0, \quad (1.16)$$

$$Y_p x_i = (a + i)(a + i + 2p)x_{i+p}, \quad Y_p x_j = 0, \quad (1.17)$$

$$L_n x_i = (a + i)x_{i+n}, \quad L_n x_j = (a + j + \frac{3}{2}n)x_{j+n}, \quad (1.18)$$

$$D_a : M_n x_k = 0, \quad (1.19)$$

$$Y_p x_i = (a + i + p)(a + i - p)x_{i+n}, \quad Y_p x_j = 0, \quad (1.20)$$

$$L_n x_i = (a + i - \frac{1}{2}n)x_{i+n}, \quad L_n x_j = (a + j + n)x_{j+n}, \quad (1.21)$$

$$A_1(\alpha) : M_n x_k = 0, \quad (1.22)$$

$$Y_p x_i = (-\frac{1}{2} + i + p)x_{i+p}, \quad Y_p x_j = 0, \quad L_n x_{\frac{1}{2}} = n(n + \alpha)x_{\frac{1}{2}+n}, \quad (1.23)$$

$$L_n x_i = (-\frac{1}{2} + i + \frac{1}{2}n)x_{i+n}, \quad L_n x_j = (-\frac{1}{2} + j + n)x_{j+n} \ (j \neq \frac{1}{2}), \quad (1.24)$$

$$A_2(\alpha) : M_n x_k = 0, \quad (1.25)$$

$$Y_p x_i = i x_{i+p}, \quad Y_p x_j = 0, \quad L_n x_{-n} = -n(n + \alpha) x_0, \quad (1.26)$$

$$L_n x_i = i x_{i+n} \quad (i \neq -n), \quad L_n x_j = (j + \frac{1}{2}n) x_{j+n}, \quad (1.27)$$

$$B_1(\alpha) : M_n x_k = 0, \quad (1.28)$$

$$Y_p x_i = 0, \quad Y_p x_j = x_{j+p}, \quad L_n x_0 = n(n + \alpha) x_n, \quad (1.29)$$

$$L_n x_i = (i + n) x_{i+n} \quad (i \neq 0), \quad L_n x_j = (j + \frac{3}{2}n) x_{j+n}, \quad (1.30)$$

$$B_2(\alpha) : M_n x_k = 0, \quad (1.31)$$

$$Y_p x_i = 0, \quad Y_p x_j = x_{j+p}, \quad L_n x_{\frac{1}{2}-n} = -n(n + \alpha) x_{\frac{1}{2}}, \quad (1.32)$$

$$L_n x_i = (-\frac{1}{2} + i - \frac{1}{2}n) x_{i+n}, \quad L_n x_j = (-\frac{1}{2} + j) x_{j+n} \quad (j \neq \frac{1}{2} - n), \quad (1.33)$$

$$C(\alpha, \alpha') : M_n x_{-n} = -2n\alpha' x_0, \quad M_n x_k = 0 \quad (k \neq -n), \quad (1.34)$$

$$Y_p x_i = i(i + 2p) x_{i+p}, \quad Y_p x_{-p} = \alpha' x_0, \quad Y_p x_j = 0 \quad (j \neq -p), \quad (1.35)$$

$$L_n x_{-n} = -n(n + \alpha) x_0, \quad L_n x_i = i x_{i+n} \quad (i \neq -n), \quad L_n x_j = (j + \frac{3}{2}n) x_{j+n}, \quad (1.36)$$

$$D(\beta, \beta') : M_n x_{\frac{1}{2}} = 2n\beta' x_{n+\frac{1}{2}}, \quad M_n x_k = 0 \quad (k \neq \frac{1}{2}), \quad Y_p x_{\frac{1}{2}} = \beta' x_{p+\frac{1}{2}}, \quad Y_p x_j = 0 \quad (j \neq \frac{1}{2}), \quad (1.37)$$

$$Y_p x_i = (-\frac{1}{2} + i + p)(-\frac{1}{2} + i - p) x_{i+n}, \quad L_n x_{\frac{1}{2}} = n(n + \beta) x_{n+\frac{1}{2}}, \quad (1.38)$$

$$L_n x_i = (-\frac{1}{2} + i - \frac{1}{2}n) x_{i+n}, \quad L_n x_j = (-\frac{1}{2} + j + n) x_{j+n} \quad (j \neq \frac{1}{2}). \quad (1.39)$$

Throughout the paper, we denote the set of all nonzero integers by \mathbb{Z}^* and that of all positive integers by \mathbb{N} .

2. Proof of Theorem 1.3(i)

The proof of Theorem 1.3 will be divided by several lemmas. Let V be an indecomposable module over $\mathcal{L}[s]$. Since M_0 and c are central elements, whose actions on V must be constants. Thus from (1.6), we can simply regard the weight space as the eigenspace of L_0 , i.e., $V = \oplus_{\lambda \in \mathbb{C}} V^\lambda$, where $V^\lambda = \{v \in V \mid L_0 v = \lambda v\}$ for $\lambda \in \mathbb{C}$.

Lemma 2.1 *Fix an $a \in \mathbb{C}$ such that $V^a \neq 0$. We have $V = \oplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$, where $V_n = V^{a+n} = \{v \in V \mid L_0 v = (a + n)v\}$ for $n \in \frac{1}{2}\mathbb{Z}$.*

Proof. For any $a \in \mathbb{C}$, denote $V(a) = \oplus_{n \in \frac{1}{2}\mathbb{Z}} V^{a+n}$. From relations (1.1)–(1.5), one can easily see that $V(a)$ is an $\mathcal{L}[s]$ -submodule of V such that $V = \oplus_{a \in \mathbb{C}/\frac{1}{2}\mathbb{Z}} V(a)$ is a direct sum of different $V(a)$. Hence $V = V(a)$ for some $a \in \mathbb{C}$. \square

Lemma 2.2 Suppose $V = \oplus_{a \in \mathbb{C}/\frac{1}{2}\mathbb{Z}} V(a)$ is an irreducible Harish-Chandra $\mathcal{L}[s]$ -module without highest and lowest weights. Then for any $i \in \mathbb{Z}^*$, $k \in \frac{1}{2}\mathbb{Z}$,

$$L_i|_{V_k} \oplus L_{i+1}|_{V_k} \oplus Y_{i+s}|_{V_k} \oplus Y_{i+1+s}|_{V_k} : V_k \rightarrow V_{k+i} \oplus V_{k+i+1} \oplus V_{k+i+s} \oplus V_{k+i+1+s} \quad (2.1)$$

is injective. In particular, by taking $i = -k$ (if $k \in \mathbb{Z}$) or $i = \frac{1}{2} - k$ (if $k \in \frac{1}{2} + \mathbb{Z}$), we obtain that $\dim V_k$ is uniformly bounded.

Proof. Suppose there exists some $v_0 \in V_k$ such that

$$L_i v_0 = L_{i+1} v_0 = Y_{i+s} v_0 = Y_{i+1+s} v_0 = 0. \quad (2.2)$$

Without loss of generality, we can suppose $i > 0$. Note that when $\ell \gg 0$, we have

$$\ell = ai + b(i+1), \quad \ell + s = a'i + b'(i+s), \quad \ell = a''(i+s) + b''(i+1+s),$$

for some $a, b, a', b', a'', b'' \in \mathbb{N}$, from this and (1.1), (1.2) and (1.4), one can easily deduce that $L_\ell, Y_{s+\ell}, M_\ell$ can be generated by $L_i, L_{i+1}, Y_{i+s}, Y_{i+1+s}$. Therefore there exists some $N > 0$ such that

$$L_\ell v_0 = Y_{s+\ell} v_0 = M_\ell v_0 = 0 \quad \text{for all } \ell \geq N.$$

The rest of the proof is exactly similar to that of [9, Proposition 2.1]. \square

Now Theorem 1.3(i) follows from Lemma 2.1.

3. Proofs of Theorem 1.3(ii) and (iii)

From now on, we shall suppose V is a module of the intermediate series over $\mathcal{L}[s]$. As in [7, 8], one sees that c acts trivially on V , so we can omit c in (1.1).

First we prove Theorem 1.3(iii), i.e., for the original Schrödinger-Virasoro Lie algebra $\mathcal{L}[\frac{1}{2}]$.

We shall first suppose that $\dim V_k = 1$ for all $k \in \frac{1}{2}\mathbb{Z}$ and both $V' = \oplus_{k \in \mathbb{Z}} V_k$ and $V'' = \oplus_{k \in \frac{1}{2} + \mathbb{Z}} V_k$ are $\mathcal{V}ir$ -modules of the form $A_{a,b}$, $a, b \in \mathbb{C}$ (later on, we shall determine all possible deformations). Therefore, we can choose a basis $\{x_k \mid k \in \frac{1}{2}\mathbb{Z}\}$ of V such that

$$Y_p x_k = f_{p,k} x_{k+p}, \quad (3.1)$$

$$M_n x_k = g_{n,k} x_{k+n}, \quad (3.2)$$

$$L_n x_k = \begin{cases} (a + k + bn)x_{k+n} & \text{if } k \in \mathbb{Z}, \\ (a + k + b'n)x_{k+n} & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases} \quad (3.3)$$

for some $a, b, b', f_{p,k}, g_{n,k} \in \mathbb{C}$, where $k \in \frac{1}{2}\mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$, $n \in \mathbb{Z}$.

Lemma 3.1 $b' = b \pm \frac{1}{2}$ or $(b, b') \in \{(0, \frac{3}{2}), (\frac{3}{2}, 0), (1, -\frac{1}{2}), (-\frac{1}{2}, 1)\}$.

Proof. If $f_{p,k} = 0$ for all $p \in \frac{1}{2} + \mathbb{Z}$, $k \in \frac{1}{2}\mathbb{Z}$, then from (1.4), we also obtain $g_{n,k} = 0$ for all $n \in \mathbb{Z}$, $k \in \frac{1}{2}\mathbb{Z}$, and so \mathcal{S} acts trivially on V . Thus there exists some $p_0 \in \frac{1}{2} + \mathbb{Z}$ and $k_0 \in \frac{1}{2}\mathbb{Z}$ such that $f_{p_0,k_0} \neq 0$. Replacing a by $a + \frac{1}{2}$ if necessary (which exchanges V' and V''), we can always suppose $f_{p_0,k_0} \neq 0$ for some $p_0 \in \frac{1}{2} + \mathbb{Z}$, $k_0 \in \mathbb{Z}$. Then from (1.2), we see that for every $p \in \frac{1}{2} + \mathbb{Z}$,

$$f_{p,k} \neq 0 \quad \text{for infinitely many } k \in \mathbb{Z}. \quad (3.4)$$

For any $m, n \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$, $k \in \mathbb{Z}$, applying

$$(p - \frac{m+n}{2})[L_m, [L_n, Y_p]] = (p - \frac{n}{2})(n + p - \frac{m}{2})[L_{m+n}, Y_p] \quad (3.5)$$

to x_k , using (3.1), (3.3) and comparing the coefficients of $x_{k+p+m+n}$, we obtain

$$\begin{aligned} & (p - \frac{m+n}{2}) \left((a+k+p+b'n)(a+k+p+n+b'm)f_{p,k} \right. \\ & \quad - (a+k+bn)(a+k+p+n+b'm)f_{p,k+n} \\ & \quad - (a+k+bm)(a+k+p+m+b'n)f_{p,k+m} \\ & \quad \left. + (a+k+bm)(a+k+m+bn)f_{p,k+m+n} \right) \\ & = (p - \frac{n}{2})(n + p - \frac{m}{2}) \left((a+k+p+b'm+b'n)f_{p,k} - (a+k+bm+bn)f_{p,k+m+n} \right). \end{aligned} \quad (3.6)$$

In the above equation, replacing m, n, k by (i) $m, m, k-m$, (ii) $-m, -m, k+m$ and (iii) $m, -m, k$, respectively, we obtain the following three equations:

$$\begin{aligned} & \left((p - \frac{m}{2})(p + \frac{m}{2})(a+k+(2b-1)m) \right. \\ & \quad \left. + (p-m)(a+k+(b-1)m)(a+k+bm) \right) f_{p,k+m} \\ & \quad - 2(p-m)(a+k+(b-1)m)(a+k+p+b'm)f_{p,k} \\ & \quad + \left((p-m)(a+k+p+(b'-1)m)(a+k+p+b'm) \right. \\ & \quad \left. - (p - \frac{m}{2})(p + \frac{m}{2})(a+k+p+(2b'-1)m) \right) f_{p,k-m} = 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \left((p+m)(a+k+p-(b'-1)m)(a+k+p-b'm) \right. \\ & \quad \left. - (p + \frac{m}{2})(p - \frac{m}{2})(a+k+p-(2b'-1)m) \right) f_{p,k+m} \\ & \quad - 2(p+m)(a+k-(b-1)m)(a+k+p-b'm)f_{p,k} \\ & \quad + \left((p + \frac{m}{2})(p - \frac{m}{2})(a+k-(2b-1)m) \right. \\ & \quad \left. + (p+m)(a+k-(b-1)m)(a+k-bm) \right) f_{p,k-m} = 0, \end{aligned} \quad (3.8)$$

$$\begin{aligned}
& (a+k+bm)(a+k+p-(b'-1)m)f_{p,k+m} \\
& - \left((a+k+p-b'm)(a+k+p+(b'-1)m) \right. \\
& \left. + (p+\frac{m}{2})(\frac{3m}{2}-p) + (a+k+bm)(a+k-(b-1)m) \right) f_{p,k} \\
& + (a+k-bm)(a+k+p+(b'-1)m)f_{p,k-m} = 0.
\end{aligned} \tag{3.9}$$

Regard (3.7)–(3.9) as a system of linear equations on the unknown variables $f_{p,k+m}$, $f_{p,k}$ and $f_{p,k-m}$. Denote by $\Delta(p, k, m)$ the coefficient determinant. By (3.4), we see that for any pairs $(p, m) \in (\frac{1}{2} + \mathbb{Z}) \times \mathbb{Z}$, there exist infinitely many integers k such that $\Delta(p, k, m) = 0$. Since $\Delta(p, k, m)$ is a polynomial on p, k, m , we obtain that

$$\Delta(p, k, m) = 0 \quad \text{for all } p \in \frac{1}{2} + \mathbb{Z}, k, m \in \mathbb{Z}. \tag{3.10}$$

It is a little lengthy but straightforward to compute (one can simply use *Mathematica* to solve a system of linear equations without problem)

$$\Delta(p, k, m) = \frac{m^6}{64} \Delta_0 (\Delta_1(a+k)p + \Delta_2 m^2 + \Delta_3 p^2),$$

where

$$\begin{aligned}
\Delta_0 &= (2b - 2b' - 1)(2b - 2b' + 1), \\
\Delta_1 &= 18(2b + 2b' - 3)(2b + 2b' - 1), \\
\Delta_2 &= 4(b + b' - 1)(-3b - 4b^2 + 4b^3 - 9b' + 4bb' + 4b^2b' + 12b'^2 - 4bb'^2 - 4b'^3), \\
\Delta_3 &= 27 - 156b + 152b^2 - 16b^3 - 16b^4 - 180b' + 328bb' - 80b^2b' \\
&\quad - 32b^3b' + 240b'^2 + 240b'^2 - 176bb'^2 - 112b'^3 + 32bb'^3 + 16b'^4.
\end{aligned}$$

It is easy to see that (3.10) holds if and only if

$$\Delta_0 = 0 \quad \text{or} \quad \Delta_1 = \Delta_2 = \Delta_3 = 0,$$

if and only if

$$b' = b \pm \frac{1}{2} \quad \text{or} \quad (b, b') \in \left\{ (0, \frac{3}{2}), (\frac{3}{2}, 0), (1, -\frac{1}{2}) \text{ or } (-\frac{1}{2}, 1) \right\}. \tag{3.11}$$

(Note that from $\Delta_1 = 0$, one obtains that $b' = \frac{3}{2} - b$ or $\frac{1}{2} - b$, then from $\Delta_2 = \Delta_3 = 0$ one can solve b .) Thus this lemma follows. \square

By replacing a by $a + \frac{1}{2}$ (then (3.4) does not necessarily hold) if necessary (which exchanges V' and V''), we can always suppose $\text{Re}(b') \geq \text{Re}(b)$ (where $\text{Re}(b)$ is the real part of the complex number b). Thus we only need to consider the cases

$$b' = b + \frac{1}{2} \quad \text{or} \quad (b, b') \in \left\{ (0, \frac{3}{2}), (-\frac{1}{2}, 1) \right\}. \tag{3.12}$$

We shall now consider all the cases given in (3.12) one by one.

Case 1 $b' = b + \frac{1}{2}$.

We need to determine $f_{p,k}$, $g_{n,k}$ defined in (3.1) and (3.2), where $n \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$ and $k \in \frac{1}{2}\mathbb{Z}$. From the relation $[Y_m, Y_p]x_k = (p - m)M_{m+p}x_k$ for $m, p \in \frac{1}{2} + \mathbb{Z}$, $k \in \frac{1}{2}\mathbb{Z}$, we only need to determine $f_{p,k}$.

Lemma 3.2 *For any $m, k \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$,*

$$(a + k + 2bp)f_{p,k+m} = (a + k + m + 2bp)f_{p,k}. \quad (3.13)$$

Proof. Using (3.7) and (3.8) to cancel $f_{p,k-m}$, one can obtain (or simply using *Mathematica*) that (3.13) holds under the following condition

$$\begin{aligned} \Delta_1(k, m) = & \left((a + k)(1 - 4b) + (1 + 6b)(b - 1)p \right) m^2 \\ & + 6(a + k)^2 p + 8(a + k)(1 - b)p^2 + 4(1 - b)p^3 \neq 0. \end{aligned} \quad (3.14)$$

Next we want to prove that (3.13) holds under the condition

$$\Delta_2(k) = (a + k)(1 - 4b) + (1 + 6b)(b - 1)p \neq 0. \quad (3.15)$$

Suppose (3.15) holds. Then $\Delta_1(k + m', m - m')$ is a polynomial on m' of degree 3 (if $b \neq \frac{1}{4}$) or of degree 2 (if $b = \frac{1}{4}$, in this case the coefficient of m'^2 in $\Delta_1(k + m', m - m')$ is $(1 + 6b)(b - 1)p + 6p = (6 - \frac{5}{8})p \neq 0$ since $p \in \frac{1}{2} + \mathbb{Z}$), and $\Delta_1(k, m')$ is a polynomial on m' of degree 2. Thus we can find some m' such that

$$(a + k + m' + 2bp)\Delta_1(k + m', m - m')\Delta_1(k, m') \neq 0,$$

in particular, condition (3.14) holds for the two triples $(k + m', m - m', p)$, (k, m', p) . Thus (3.13) holds for these two triples, and one has

$$\begin{aligned} (a + k + 2bp)f_{p,k+m} &= \frac{a + k + 2bp}{a + k + m' + 2bp}(a + (k + m') + 2bp)f_{p,(k+m')+(m-m')} \\ &= \frac{a + k + 2bp}{a + k + m' + 2bp}(a + (k + m') + (m - m') + 2bp)f_{p,k+m'} \\ &= \frac{a + k + m + 2bp}{a + k + m' + 2bp}(a + k + 2bp)f_{p,k+m'} \\ &= (a + k + m + 2bp)f_{p,k}. \end{aligned} \quad (3.16)$$

Thus (3.13) holds under condition (3.15). Now for any p, k, m , choose m' such that

$$(a + k + m' + 2bp)\Delta_2(k + m') \neq 0. \quad (3.17)$$

This follows by noting that if $b \neq \frac{1}{4}$ then $\Delta_2(k + m')$ is a polynomial on m' of degree 1, and if $b = \frac{1}{4}$ then $\Delta_2(k + m') = (1 + 6b)(b - 1)p = -\frac{15}{8}p \neq 0$ (since $p \in \frac{1}{2} + \mathbb{Z}$). Now (3.17)

implies that we have all equalities of (3.16) except the last equality. But the last equality also follows by writing $f_{p,k}$ as $f_{p,(k+m')+(-m')}$ and using (3.13) with the pair (k, m) being replaced by $(k + m', -m')$ (condition (3.15) holds for this pair). \square

For any $m \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$, $k \in \frac{1}{2}\mathbb{Z}$, applying $[L_m, Y_p] = (p - \frac{m}{2})Y_{p+m}$ to x_k and comparing the coefficients of x_{k+m+p} , one has

$$\begin{cases} (a + k + bm)f_{p,m+k} - (a + k + p + b'm)f_{p,k} = \frac{m-2p}{2}f_{m+p,k} & \text{if } k \in \mathbb{Z}, \\ (a + k + b'm)f_{p,m+k} - (a + k + p + bm)f_{p,k} = \frac{m-2p}{2}f_{m+p,k} & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases} \quad (3.18)$$

Taking $m = p - n$ in (3.18) and using $b' = b + \frac{1}{2}$, one has

$$(a + k + bp - bn)f_{n,p-n+k} - (a + k + n + bp - bn + \frac{p-n}{2})f_{n,k} = \frac{p-3n}{2}f_{p,k}. \quad (3.19)$$

Using (3.13), one has

$$(a + k + 2bn)f_{n,k+p-n} = (a + k + p - n + 2bn)f_{n,k}. \quad (3.20)$$

Then using (3.19) together with (3.13) and (3.20), we obtain

$$(p - 3n)(a + k + 2bp)((a + k + 2bn)f_{p,k+m} - (a + k + m + 2bp)f_{n,k}) = 0. \quad (3.21)$$

Letting $m = j - k$ in (3.21), one has

$$(p - 3n)(a + k + 2bp)((a + k + 2bn)f_{p,j} - (a + j + 2bp)f_{n,k}) = 0. \quad (3.22)$$

Lemma 3.3 *For any $k, j \in \mathbb{Z}$, $n, p \in \frac{1}{2} + \mathbb{Z}$, one has*

$$(a + k + 2bn)f_{p,j} = (a + j + 2bp)f_{n,k}. \quad (3.23)$$

Furthermore, for any $j \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$, $f_{p,j}$ can be written as

$$f_{p,j} = (a + j + 2bp)f_0, \quad (3.24)$$

for some $f_0 \in \mathbb{C}$.

Proof. Equation (3.24) follows from (3.23) by fixing n_0, k_0 such that $a + k_0 + bn_0 \neq 0$ and letting $f_0 = \frac{f_{n_0,k_0}}{a+k_0+bn_0}$. It remains to prove (3.23). If $(p - 3n)(a + k + 2bp) \neq 0$, then (3.23) follows from (3.22). Next suppose $(p - 3n)(a + k + 2bp) = 0$. Choose $n_0 \in \frac{1}{2} + \mathbb{Z}$ and $k_0 \in \mathbb{Z}$ such that

$$(p - 3n_0)(a + k_0 + 2bp) \neq 0 \quad \text{and} \quad (n - 3n_0)(a + k_0 + 2bn) \neq 0.$$

Then by (3.22), $(a + k_0 + 2bn_0)f_{p,j} = (a + j + 2bp)f_{n_0,k_0}$ and also $(a + k + 2bn)f_{n_0,k_0} = (a + k_0 + 2bn_0)f_{n,k}$. Thus (3.23) also holds. This proves the lemma. \square

As for the case $k \in \frac{1}{2} + \mathbb{Z}$, we have the following lemma.

Lemma 3.4 For any $m, k, p \in \frac{1}{2} + \mathbb{Z}$,

$$f_{p,k} = d_0. \quad (3.25)$$

Proof. First suppose $b \neq 0$. We want to prove

$$f_{\frac{1}{2},k} = d_0 \quad \text{for } k \in \frac{1}{2} + \mathbb{Z}. \quad (3.26)$$

Applying $[L_1, Y_{\frac{1}{2}}] = 0$ to x_k for $k \in \frac{1}{2} + \mathbb{Z}$ and using $b' = b + \frac{1}{2}$, we obtain

$$(a + k + b + \frac{1}{2})f_{\frac{1}{2},k} - f_{\frac{1}{2},k+1}(a + k + b + \frac{1}{2}) = 0. \quad (3.27)$$

If $a + b \notin \mathbb{Z}$, then $a + k + b + \frac{1}{2} \neq 0$ for all $k \in \frac{1}{2} + \mathbb{Z}$, thus (3.27) implies $f_{\frac{1}{2},k}$ is a constant, denoted by d_0 . Thus (3.26) holds in this case. Assume that $a + b \in \mathbb{Z}$. If necessary, by shifting the index k of x_k by an integer (which does not change V', V''), we can suppose $a + b = 0$. Then (3.27) shows

$$f_{\frac{1}{2},k} = \begin{cases} f_{\frac{1}{2},\frac{1}{2}} & \text{if } k > 0, \\ f_{\frac{1}{2},-\frac{1}{2}} & \text{if } k < 0. \end{cases} \quad (3.28)$$

Applying $[L_{-1}, Y_{\frac{1}{2}}] = Y_{-\frac{1}{2}}$ to x_k , we obtain $f_{-\frac{1}{2},k} = (a + k + \frac{1}{2} - b)f_{\frac{1}{2},k} - f_{\frac{1}{2},k-1}(a + k - (b + \frac{1}{2}))$, which together with (3.28) gives (using $a = -b$)

$$f_{-\frac{1}{2},k} = \begin{cases} f_{\frac{1}{2},\frac{1}{2}} & \text{if } k > \frac{1}{2}, \\ (1 - 2b)f_{\frac{1}{2},\frac{1}{2}} + 2bf_{\frac{1}{2},-\frac{1}{2}} & \text{if } k = \frac{1}{2}, \\ f_{\frac{1}{2},-\frac{1}{2}} & \text{if } k \leq -\frac{1}{2}. \end{cases} \quad (3.29)$$

Applying $[L_1, Y_{-\frac{1}{2}}] = -Y_{\frac{1}{2}}$ to $x_{\frac{1}{2}}$, using (3.28) and (3.29), we obtain

$$(1 - 2b)f_{\frac{1}{2},\frac{1}{2}} + 2bf_{\frac{1}{2},-\frac{1}{2}} = -f_{\frac{1}{2},\frac{1}{2}} = (a + \frac{1}{2} - \frac{1}{2} + b)f_{-\frac{1}{2},\frac{1}{2}} - f_{-\frac{1}{2},\frac{3}{2}}(a + \frac{1}{2} + b + \frac{1}{2}),$$

which implies $f_{\frac{1}{2},\frac{1}{2}} = f_{\frac{1}{2},-\frac{1}{2}}$ since $a = -b \neq 0$. This together with (3.28) gives (3.26).

Now for any $\frac{2}{3} \neq p \in \frac{1}{2} + \mathbb{Z}$, by replacing (n, p) by $(p - \frac{1}{2}, \frac{1}{2})$ in (1.2) and applying it to x_k , one can easily obtain $f_{p,k} = d_0$. If $p = \frac{2}{3}$, choosing some $n \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$ such that $n + p = \frac{2}{3}$ and $p - \frac{n}{2} \neq 0$, we can again obtain $f_{\frac{2}{3},k} = d_0$. This proves (3.25) for the case $b \neq 0$.

Now suppose $b = 0$. Similar to the proof of (3.13), for any $m, n \in \mathbb{Z}$, $k, p \in \frac{1}{2} + \mathbb{Z}$, one has

$$f_{p,k+m} = f_{p,k}, \quad (3.30)$$

under the condition (noting that $b = 0$)

$$\nabla(k, m) = \nabla_2(k)m^2 + \nabla_0(k) \neq 0, \quad (3.31)$$

where

$$\begin{aligned} \nabla_2(k) &= (a+k)^2 + 3(a+k)p + 2p^2, \\ \nabla_0(k) &= -6(a+k)^3p - 10(a+k)^2p^2 - 6(a+k)p^3 - 2p^4. \end{aligned}$$

From (3.31), one can easily find some m' such that (since the left-hand side of (3.32) is a polynomial on m' of degree 8)

$$\nabla(k+m', m-m')\nabla(k+m', -m') \neq 0. \quad (3.32)$$

Then (3.30) holds for the pairs $(k+m', m-m')$, $(k+m', -m')$, which forces

$$f_{p,k+m} = f_{p,(k+m')+(m-m')} = f_{p,(k+m')} = f_{p,(k+m')+(-m')} = f_{p,k}.$$

Thus the lemma follows. \square

For any $m, n \in \frac{1}{2} + \mathbb{Z}$, $k \in \mathbb{Z}$, applying $[Y_m, Y_n] = (n-m)M_{m+n}$ to x_k and comparing the coefficients of x_{k+m+n} , one has

$$(n-m)(g_{m+n,k} - 2bd_0f_0) = 0. \quad (3.33)$$

Taking $n = p - m$ with $p - 2m \neq 0$ in (3.33), one immediately obtains

$$g_{p,k} = 2bd_0f_0 \quad \text{for all } k, p \in \mathbb{Z}. \quad (3.34)$$

Now applying $[Y_m, Y_n] = (n-m)M_{m+n}$ to x_k for $k \in \frac{1}{2} + \mathbb{Z}$, and comparing the coefficients of x_{k+m+n} , one has

$$(n-m)(g_{m+n,k} - (1-2b)d_0f_0) = 0. \quad (3.35)$$

Thus as in (3.34), we obtain

$$g_{p,k} = (1-2b)d_0f_0 \quad \text{for all } p \in \mathbb{Z}, k \in \frac{1}{2} + \mathbb{Z}. \quad (3.36)$$

By now, we have obtained the following relations

$$f_{p,k} = \begin{cases} d_0 & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \\ (a+k+2bp)f_0 & \text{if } k \in \mathbb{Z}, \end{cases} \quad g_{n,k} = \begin{cases} (1-2b)d_0f_0 & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \\ 2bd_0f_0 & \text{if } k \in \mathbb{Z}. \end{cases} \quad (3.37)$$

For any $p \in \frac{1}{2} + \mathbb{Z}$, $n, k \in \mathbb{Z}$, applying $[Y_p, M_n] = 0$ to x_k and comparing the coefficients of x_{k+n+p} , one has

$$2(a+n+k+2bp)bd_0f_0^2 = (a+k+2bp)(1-2b)d_0f_0^2. \quad (3.38)$$

Comparing the coefficients of n^1 and n^0 , we obtain $bd_0f_0^2 = (1 - 2b)d_0f_0^2 = 0$, which implies

$$d_0f_0 = 0. \quad (3.39)$$

Since V is indecomposable, we obtain

$$d_0 = 0, f_0 \neq 0 \quad \text{or} \quad d_0 \neq 0, f_0 = 0, \quad (3.40)$$

and the second equation of (3.37) can be rewritten as

$$g_{n,k} = 0 \quad \text{for all } n \in \mathbb{Z}, k \in \frac{1}{2}\mathbb{Z}. \quad (3.41)$$

For the first case of (3.40), by rescaling x_k for $k \in \mathbb{Z}$ (and keeping $x_k, k \in \frac{1}{2} + \mathbb{Z}$ unchanged), we can suppose

$$f_0 = 1. \quad (3.42)$$

Similarly, for the last case of (3.40), we can suppose

$$d_0 = 1. \quad (3.43)$$

Then we get the two types of irreducible modules of intermediate series over \mathcal{L} , denoted by $A_{a,b}, B_{a,b}$ with the basis $\{x_k \mid k \in \frac{1}{2}\mathbb{Z}\}$ and the actions given by (1.10)–(1.12) and (1.13)–(1.15) respectively.

Now we consider all possible deformations of the representation $A_{a,b}$. In order for $A_{a,b}$ to have a nontrivial deformation, it is necessary that $A_{a,b}$ is decomposable. Thus we only need to consider the following 4 subcases.

Subcase 1.1 $a \in \mathbb{Z}, b = 1$.

In this case, by shifting the index k , we can suppose $a = 0, b = 1$. Then $x_k, 0 \neq k \in \frac{1}{2}\mathbb{Z}$ span the only proper submodule. Thus the possible deformations are the actions of M_n, Y_p, L_n on x_0 .

In order to get nontrivial deformation, we firstly remark that the actions L_n on x_0 must have deformation, otherwise from our computations above, the actions of M_n, Y_p would not have any deformation. Furthermore, the deformation of the action of L_n on x_0 can be seen in (1.8). Thus according to (1.10)–(1.12), we have

$$M_n x_k = 0 \quad \text{for any } k \in \frac{1}{2}\mathbb{Z}^*, \quad (3.44)$$

$$Y_p x_k = \begin{cases} 0 & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \\ (k + 2p)x_{k+p} & \text{if } k \in \mathbb{Z}^*, \end{cases} \quad (3.45)$$

$$L_n x_k = \begin{cases} (k + \frac{3}{2}n)x_{k+n} & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \\ (k + n)x_{k+n} & \text{if } k \in \mathbb{Z}^*, \\ n(n + \alpha)x_n & \text{if } k = 0, \end{cases} \quad (3.46)$$

for some $\alpha \in \mathbb{C}$, where $p \in \frac{1}{2} + \mathbb{Z}$, $n \in \mathbb{Z}$. And we suppose

$$M_n x_0 = f_n x_n, \quad Y_p x_0 = g_p x_p, \quad n \in \mathbb{Z}, \quad p \in \frac{1}{2} + \mathbb{Z}, \quad (3.47)$$

for some $f_n, g_p, h_n \in \mathbb{C}$. For any $n \in \mathbb{Z}^*$, $p \in \frac{1}{2} + \mathbb{Z}$, applying $[L_n, Y_p] = (p - \frac{n}{2})Y_{n+p}$ to x_0 and comparing the coefficients of x_{n+p} , we obtain

$$(p + \frac{3n}{2})g_p - n(n + \alpha)(n + 2p) = (p - \frac{n}{2})g_{n+p}. \quad (3.48)$$

Letting $n = 2p$ in (3.48), we obtain $g_p = 2p(2p + \alpha)$, and also $g_{n+p} = 2(n + p)(2(n + p) + \alpha)$. Using them in (3.48), we obtain a contradiction, i.e., in this case $A_{a,b}$ has no deformation.

Subcase 1.2 $a \in \mathbb{Z}$, $b = 0$.

Similarly, we can suppose $a = b = 0$ by shifting the index k . Then $\mathbb{F}x_0$ is the only proper submodule (thus the deformation of the action of L_n on x_{-n} can be seen in (1.9)). Hence in order to obtain the identities listed in (1.25)–(1.27), we only need to compute the complex numbers g_p and f_n defined by

$$Y_p x_{-p} = g_p x_0, \quad M_n x_{-n} = f_n x_0 \quad \text{for all } n \in \mathbb{Z}, \quad p \in \frac{1}{2} + \mathbb{Z}. \quad (3.49)$$

For any $n \in \mathbb{Z}^*$, $p \in \frac{1}{2} + \mathbb{Z}$, applying $[L_n, Y_p] = (p - \frac{n}{2})Y_{n+p}$ to x_{-n-p} and comparing the coefficients of x_0 , we obtain $g_p = 0$ for any $p \in \frac{1}{2} + \mathbb{Z}$. Then

$$Y_p x_{-p} = 0 \quad \text{for any } p \in \frac{1}{2} + \mathbb{Z}. \quad (3.50)$$

By (3.50) and using $[Y_p, Y_{n-p}]x_{-n} = (n - 2p)M_n x_{-n}$, one has

$$M_n x_{-n} = 0 \quad \text{for any } n \in \mathbb{Z}. \quad (3.51)$$

Then we get a deformation of $A_{a,b}$, denoted by $A_2(\alpha)$ with the relations given by (1.25)–(1.27).

Subcase 1.3 $a \in \frac{1}{2} + \mathbb{Z}$, $b = \frac{1}{2}$.

Similarly, shifting the index k , one can assume $a = -\frac{1}{2}$, $b = \frac{1}{2}$. Then $x_k, \frac{1}{2} \neq k \in \frac{1}{2}\mathbb{Z}$, span the only submodule. Using the same techniques, we can obtain

$$M_n x_{\frac{1}{2}} = Y_p x_{\frac{1}{2}} = 0 \quad \text{for all } n \in \mathbb{Z}, \quad p \in \frac{1}{2} + \mathbb{Z}. \quad (3.52)$$

Then we can obtain a deformation, denoted by $A_1(\alpha)$ with the relations given by (1.22)–(1.24) (in fact $A_1(\alpha)$ is isomorphic to the dual module of $A_2(\alpha')$ for some $\alpha' \in \mathbb{C}$).

Subcase 1.4 $a \in \frac{1}{2} + \mathbb{Z}$, $b = -\frac{1}{2}$.

In this case, by shifting the index k , we can suppose $a = -\frac{1}{2}$, $b = -\frac{1}{2}$. Then by (1.10)–(1.12), one has

$$M_n x_k = 0 \quad \text{for any } k \in \frac{1}{2}\mathbb{Z}, n \neq \frac{1}{2} - n, \quad (3.53)$$

$$Y_p x_k = \begin{cases} 0 & \text{if } k \in \frac{1}{2} + \mathbb{Z}, k \neq \frac{1}{2} - p, \\ (-\frac{1}{2} + k - p)x_{k+p} & \text{if } k \in \mathbb{Z}, \end{cases} \quad (3.54)$$

$$L_n x_k = \begin{cases} (-\frac{1}{2} + k)x_{k+n} & \text{if } k \in \frac{1}{2} + \mathbb{Z}, k \neq \frac{1}{2} - n, \\ (-\frac{1}{2} + k - \frac{1}{2}n)x_{k+n} & \text{if } k \in \mathbb{Z}, \\ -n(n + \alpha)x_{\frac{1}{2}} & \text{if } k = \frac{1}{2} - n, \end{cases} \quad (3.55)$$

where $p \in \frac{1}{2} + \mathbb{Z}$, $n \in \mathbb{Z}$, $\alpha \in \mathbb{C}$.

In order to determine all possible deformations, we suppose

$$M_n x_{\frac{1}{2}-n} = f_n x_{\frac{1}{2}}, \quad Y_p x_{\frac{1}{2}-p} = g_p x_{\frac{1}{2}} \quad \text{for all } n \in \mathbb{Z}, p \in \frac{1}{2} + \mathbb{Z}. \quad (3.56)$$

For any $n \in \mathbb{Z}^*$, $p \in \frac{1}{2} + \mathbb{Z}$, applying $[L_n, Y_p] = (p - \frac{n}{2})Y_{n+p}$ to $x_{\frac{1}{2}-n-p}$ and comparing the coefficients of $x_{\frac{1}{2}}$, we obtain

$$(p + \frac{3n}{2})g_p + n(n + 2p)(n + \alpha) = (p - \frac{n}{2})g_{n+p}. \quad (3.57)$$

Letting $n = 2p$ in (3.57), we obtain $g_p = -2p(2p + \alpha)$. As in Subcase 1.1, we obtain a contradiction with (3.57). So in this case $A_{a,b}$ has no deformation either.

Similarly, we can also give all possible deformations of the representation $B_{a,b}$ denoted by $B_1(\alpha)$ and $B_2(\alpha)$, whose module structures are listed in (1.28)–(1.33).

Case 2 $b = 0$, $b' = \frac{3}{2}$.

Regardless of their deformations, the module structures over \mathcal{L} corresponding to $(b, b') = (0, \frac{3}{2})$ can be determined as those corresponding to the case $(b, b') = (1, \frac{3}{2})$, i.e., $A_{1, \frac{3}{2}}$ and $B_{1, \frac{3}{2}}$, whose module structures have been listed in (1.10)–(1.15) explicitly. Using the same techniques, we obtain an indecomposable module of intermediate series over \mathcal{L} , denoted by C_a , given by (1.16)–(1.18) and its possible deformation, denoted by $C(\alpha, \alpha')$ and given by (1.34)–(1.36).

Case 3 $b = -\frac{1}{2}$, $b' = 1$.

Regardless of their deformations, the module structures over \mathcal{L} corresponding to $(b, b') = (-\frac{1}{2}, 1)$ can be determined as those corresponding to the case $(b, b') = (-\frac{1}{2}, 0)$, i.e., $A_{-\frac{1}{2}, 0}$ and

$B_{-\frac{1}{2},0}$, whose module structures have been listed in (1.10)–(1.15) explicitly. Using the same techniques, we obtain an indecomposable module of intermediate series over \mathcal{L} , denoted by D_a , given by (1.19)–(1.21) and its possible deformation, denoted by $D(\beta, \beta')$ and given by (1.37)–(1.39).

This completes the proof of Theorem 1.3(iii).

Finally we consider the case $\mathcal{L}[0]$. In this case, $V_k = 0$ for $k \in \frac{1}{2} + \mathbb{Z}$, and so V is \mathbb{Z} -graded. As the proof above (now all indices k, p, n are in \mathbb{Z}), we still can suppose (3.1), (3.2) and the first case of (3.3). Thus we again have (3.6) with b' being b . But the proof of Lemma 3.1 shows that $b' = b$ is impossible, i.e., \mathcal{S} must act trivially on V . This proves Theorem 1.3(ii). \square

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